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ABSTRACT

This paper provides an historical perspective on the possible roles of cognitive technologies in thinking as "reorganizers" of the mind. It suggests that by understanding the transformational roles of advanced technologies for mathematical thinking, positive contributions can be made to research and practice on the use of computers in mathematics education. A heuristic taxonomy of seven functions that mathematics educational technologies can serve to promote the aims of mathematical thinking is proposed. Two types of functions are distinguished: (1) "purpose" functions, that may affect whether students choose to think mathematically and (2) "process" functions that may support the component mental activities of mathematical thinking. Definitions and examples of computer software are used throughout the document to illustrate the seven functions. (TW)

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COGNITIVE TECHNOLOGIES FOR MATHEMATICS EDUCATION

Roy D. Pea

Technical Report No. 37

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COGNITIVE TECHNOLOGIES FOR MATHEMATICS EDUCATION*,**

Roy D. Pea

My plan in this paper is first to develop an historical perspective on roles for cognitive technologies in thinking as reorganizers rather than amplifiers of mind. It is in this context, I will suggest, that we can best understand the transformational roles of advanced technologies for mathematical thinking and education, and thereby contribute productively to research and practice on computers in mathematics education. Computers are doing far more than making it easier to do what we are doing. It is in terms of this historical context that the drastic reformulations of aims and methods of mathematics education computers have wrought makes sense. I then propose a heuristic taxonomy of seven functions that mathematics educational technologies can serve to promote the aims of mathematical thinking. Two types of functions are distinguished: purpose functions that may affect whether students choose to think mathematically; and process functions that may support the component mental activities of mathematical thinking. My hope is that the functions distinguished with these two categories express transcendent features of cognitive technologies helpful for students learning to think mathematically, and that they can be used retroactively to assess existing software, and proactively to guide software development efforts. Definitions and examples of software are used throughout to illustrate the seven functions.

Themes of Change in the Aims and Methods of Mathematics Education

There is no question but that information technologies, in particular the computer, have radical implications for and are already changing the methods, and thus the practices, of mathematics education. But, perhaps more importantly, we are also coming to see that they are changing its aims, and thereby what we consider the goals of mathematical "understanding" and "thinking" to which our educational processes are directed.

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**I have had opportunity to discuss some of these issues with Wally Feurzeig, Ron Mawby, Andrea Pettito, Al Schoenfeld, Judah Schwartz, Kurt VanLehn, Ron Wenger, Jim Wilson, and members of the Sloan Workgroup on Cognitive Science and Mathematics Education. Jim Kaput, in particular, has influenced greatly this account of how drastically information technologies are changing and will continue to change our curriculum topics and educational goals in mathematics. None of those mentioned above necessarily subscribe to the taxonomy and arguments presented here.

More so than in any other curriculum field, mathematics educators, represented by such organizations as NCTM, are fundamentally rethinking their aims and means. In particular, mathematics activities can now take their meaning in broadly different contexts of application than ever before. The reason for this is the widespread availability of powerful mathematical tools that simplify numerical and symbolic calculations, graphing, and modeling, and many of the mental operations involved in mathematical thinking. For example, many classrooms now have available programmable calculators, computer languages, simulation and modeling languages, spreadsheets, algebraic equation solvers such as TK!Solver, symbolic manipulation packages, and software for data analysis and graphing. The drudgery of remembering and practicing cumbersome algorithms is now often supplanted by the quite different activities of selecting appropriate computer programs and data entry. And the central role of mathematical thinking in mathematics education is now receiving more attention, both among educators and in the research community (e.g., Schoenfeld, 1985a; Silver, 1985): "You understand how to think mathematically when you are resourceful, flexible, and efficient in your ability to deal with new problems in mathematics" (Schoenfeld, 1985a, p. 2).

Why have these revolutionary changes occurred? And how can we think about these changes as a guide to the design, testing, and use of new technologies by educators to enhance the processes of mathematics education (and our understanding of how it occurs)? In other words, are there any beacons to help light the paths to future effectiveness?

Historical Approach to Mental Roles for Cognitive Technologies

I will argue that an historical approach will serve us well as we think about how the powers of information technologies can serve mathematics education and research. Is there really something new about computers, or are we just better supported in doing the same mathematical problem solving and education to which we are accustomed? Long before the advent of computers, there were remarkable extensions of human intelligence through the use of technical instruments. This section is devoted to providing a broader historical frame for thinking about the transformational roles of computers as tokens of a type--cognitive technologies--and for illuminating the structure and loci of their potentials as tools of mentation. I take as axiomatic that intelligence is not a quality of the mind alone, but a product of the relation between mental structures and the tools of the intellect provided by the culture (Bruner, 1966; Cole & Griffin, 1980; Luria, 1976, 1979; Olson, 1976, 1985; Olson & Bruner, 1974; Vygotsky, 1962, 1978). Let us call these tools "cognitive technologies."

Contrast for a moment what it meant to learn mathematics with a chalk and board, where one erased after each problem, with using paper and pencil, where one can save and inspect one's work. This example reveals that under the broad rubric of cognitive technologies for mathematics, we must include entities as diverse as the chalk and board, the pencil and paper, the computer and screen, and the symbol systems within which mathematical discoveries have been made,

and that have led to the creation of new symbol systems. Each has transformed how mathematics can be done, and how mathematics education could be accomplished. If time allowed, it would be revealing to describe how mathematics and mathematics education was accomplished quite differently with each medium.

A cognitive technology is provided by any medium that helps transcend the limitations of the mind (e.g., attention to goals, short-term memory span) in activities of thinking, learning, and problem solving. These technologies have had remarkable consequences for the varieties of intelligence, the functions of human thinking, and the intellectual achievements of prior history (e.g., Cassirer, 1944; Goodman, 1976). They include symbol systems generally, including writing systems, logics, mathematical notation systems, models, theories, film and other pictorial media, and now symbolic computer languages. The technologies that have received perhaps the most attention as cognitive tools are written language (Goody, 1977; Greenfield, 1972; Olson, 1977; Ong, 1982; Scribner & Cole, 1981) and systems of mathematical notation, such as algebra or calculus (Cassirer, 1910, 1957; Kaput, 1985, in press; Kline, 1972).

A common feature of all these cognitive technologies is that they make external the intermediate products of thinking (e.g., outputs of component steps in solving a complex algebraic equation), which can then be analyzed, reflected upon, and discussed. Transient and private thought processes subject to the distortions and limitations of attention and memory are "captured" and embodied in a communicable medium that persists, providing material records that can become objects of analysis in their own right--conceptual building blocks rather than shifting sands. Vygotsky (1978) heralded these tools as the "extracortical organizers of thought" since they help organize thinking but outside the physical confines of the brain.

We are now seeing, in ways to be described throughout this paper, how computers are an especially potent species of cognitive technologies for learning to think mathematically, since they can operate not only with numbers, but with symbols--the fundamental currency of human thought. Computers are universal machines for storing and dynamically manipulating symbols--images, words, numbers--which appear to serve as the currency of human thought. Capable of real-time programmable interactions with human users, computers may provide the most extraordinary cognitive technology yet devised. But what can we learn from the history of noncomputer-based cognitive technologies that will inform our current inquiries?

Cognitive technologies, such as written languages, are commonly thought of as "cultural amplifiers" of the intellect, to use Bruner's (1966) influential phrase. They are viewed as a cultural means for empowering human cognitive capacities. Greenfield and Bruner (1969) observed that cultures with technologies such as written language and mathematical formalisms will "push cognitive growth better, earlier, and longer than others" (p. 654). We find similar upbeat predictions for computer technologies embodied in a widespread belief that they

will inevitably and profoundly amplify human mental powers (Pea & Kurland, 1984).

This amplifier metaphor for cognitive technologies has led to many research programs, particularly on the cognitive consequences of literacy and schooling (e.g., on formal logical reasoning) in the several decades since Bruner and his colleagues published Studies in Cognitive Growth (e.g., Greenfield, 1972; Olson, 1976; Scribner & Cole, 1981). The amplifier metaphor continues in contemporary work on electronic technologies by John Seeley Brown of Xerox PARC, who in a recent paper describes his prototype software systems for writing and doing mathematics as "idea amplifiers" (Brown, 1984a). For example, AlgebraLand, created by Brown and colleagues (Brown, 1984b), is a software program in which students are freed from hand calculations associated with executing different algebraic operations, and focus on high-level problem-solving strategies which they select for the computer to perform. AlgebraLand is said to enable students to explore the problem space faster as they learn equation-solving skills. Although quantitative metrics, such as the efficiency and speed of learning, may truly describe changes that occur in problem solving with electronic tools, it can be shown that more profound changes--as I later describe for the AlgebraLand example--may be missed if we confine ourselves to the amplification perspective.

There is a different tradition in the study of cognitive technologies that may be characterized as cultural-historical. This perspective is most familiar to psychologists and educators today in the influential work of Vygotsky (1978). He offered an account of the development of higher mental functions, such as planning and numerical reasoning, as based upon the "internalization" of self-regulatory activities that first take place in social interactions between children and adults. The historical roots of Vygotsky's orientation, to be described, provide an illuminating framework for the roles of computer technologies in mathematical thinking and learning. Influenced by the writings of Vico, Spinoza, and Hegel, Marx and Engels developed a novel and powerful theory of society now described as "historical" or "dialectical materialism." Human nature, in this view, rather than being a product of environmental forces is of our own making as a society, and is continually "becoming." Humankind is reshaped through a dialectic, or "conversation," of reciprocal influences: Our productive activities change the world, thereby changing the ways in which the world can change us. By shaping nature and how our interactions with it are mediated, we change ourselves. As the biologist Stephen Jay Gould (1980) observed, such "cultural evolution," in contrast to Darwinian biological evolution, is defined by transmission of skills, knowledge, and behavior through learning across generations, and has been our nature-transcendent innovation as a species.

In this cultural-historical perspective, labor is seen as the mediating factor between humans and nature. By creating and using physical instruments (such as machinery) that mediate in less and less direct ways our interactions with nature, we come to reshape human nature. Note how a change in the instruments of work (e.g., a plow rather than

the hand) changes the functional organization, or system characteristics, of humans' relation to work--not only is the work accomplished faster, but the tasks themselves differ.

In efforts to integrate accounts of individual and cultural changes, the Soviet theorists Vygotsky (1962, 1978) and Luria (1976, 1979) generalized the historical materialism that Marx and Engels developed for physical instruments, and applied it to an historical analysis of symbolic tools (such as written language) that serve as instruments for redefining culture and human nature. Vygotsky recognized that "mental processes always involve signs, just as action on the environment always involves physical instruments (if only a human hand)" (Scribner & Cole, 1981, p. 8). A similar instrumental and dialectical perspective is reflected by recent studies of the "child as a cultural invention" (Kessel & Siegel, 1983; Kessen, 1979; White, 1983). Take for instance, Wartofsky's (1983) statement of the shift in perspective:

> Children are, or become, what they are taken to be by others, and what they come to take themselves to be, in the course of their social communication and interactions with others. In this sense, I take "child" to be a social and historical kind, rather than a natural kind, and therefore also a constructed kind rather than one given, so to speak, by nature in some fixed or essential form. (p. 190)

Applied to mathematics education, this sociohistorical perspective highlights not the constancy of the mathematical understandings of which children "are capable" at particular ages, but how what we take for granted as "limits" get redefined by the child's use of new cognitive technologies for learning and doing mathematics. Similarly, Cole and Griffin (1980) noted how symbolic technologies qualitatively change the structure of the functional system for such mental activities as problem solving or memory. By contrast, use of the term "amplify" means to make more powerful; to amplify in the scientific sense "refers rather specifically to the intensification of a signal (acoustic, electronic) that does not undergo change in its basic structure " (Cole & Griffin, 1980, p. 349). As such, the amplifier metaphor for roles of technologies in mathematical thinking leads one to unidimensional, quantitative theorizing about the effects of cognitive technologies. As evidence of such tendencies, consider how a pencil can be thought of as amplifying the power of a sixth grader's memory for a long list of words when only the outcome of the list length is considered. But it would be distortive to go on to say that the mental process of remembering that led to the outcome was amplified by the pencil, since "remembering" in the two cases refers to two qualitatively different activities. The pencil did not amplify a fixed mental capacity called memory; it restructured the functional system for remembering, and thereby led to a more powerful outcome (at least for the purpose of remembering more items). Similar preoccupations with amplification lead to measures of the enhancements in basic mathematics facts learning with software over print media, rather than the fundamental changes in arithmetical thinking that accompany uses

of programmable calculator functions (Conference Board of the Mathematical Sciences, 1983; Fey, 1984; NSB, 1983).

Olson (1976) makes similar arguments about restructurings of thinking processes created through written language. For example, logical analysis of arguments for consistency-contradiction becomes possible because memory limitations for oral language are mitigated, and print--rather than oral narrative--provides a means to store and communicate cultural knowledge. It is important to note that the specific restructurings of cognitive technologies are rarely predictable; they have emergent properties that come to be discovered only through experimentation. We can expect the same in mathematics education.

What this means is that technologies do not simply amplify, like a radio amplifier, the learner's mental powers or make faster and more efficient a learning process toward already-decided-upon educational goals. The standard one-way mental image of the cognitive effects of computer use is a child seated at a computer terminal, with certain changes transpiring in that child as a direct function of his or her interaction with the machine. It is also a seductive setting for the researcher, since there may be fewer variables to measure in the learning context. But since the technologies change the system of thinking activities in which the technologies play a role, their effects are much more complex and often indirect.

I therefore espouse a quite different story about the cognitive effects of computers than the one-way line. But it is one consistent with questions spurred by two-way reasoning that other mathematics educators and researchers (e.g., Kaput 1985, in press) are asking, such as: What are the new things you can do with technologies that you could not do before, or that weren't practical to do? and, What totally new things--once you begin to use the technology--do you realize it might be possible to do? By "two-way reasoning," I mean that it is not just that computers affect people, but that people affect computers. This is true in two senses. In one sense, we all affect computers and the learning opportunities they afford students by how we interpret them and what we define as appropriate practices with them; as these interpretations change over time, we importantly change the effects the computers could have by changing what we do with them. (Consider how we began in schools with drill-and-practice and computer-literacy activities, and now emphasize uses of computers as tools such as word processors, spreadsheets, and database management systems.) In another sense, we affect computers when we study their use, reflect on what we see happening, and then act to change it in ways we prefer or see as necessary to get the effects we want. And such software engineering is fundamentally a two-way conversational process between humans and machines. We define the educational goals (either tacitly or explicitly) and then create the learning activities we want the software to support toward these goals. We experiment and test, experiment and test, until we are satisfied--which we tend never to be. This is why the two-way reasoning "spirals"--it is the spiral process toward the unknown. Through experimentation, new goals and new

ideas for learning activities emerge. And so it goes on, as we create our own histories through remaking the tools we learn and think with, and our goals for their use. We can see that the horizons of future mathematics education and its goals are under pressure for definition by new technologies. To understand the reciprocal influences that occur and that are possible, we must acknowledge that these new educational goals emerge experimentally.

Cognitive Technologies in Mathematics Education

How does the idea of cognitive technologies relate to mathematics education? A few historical notes prepare the stage. We may recall Ernst Mach's emphasis, in his seminal work on the science of mechanics earlier in this century, that mathematics should have the purpose of saving mental effort. Thus, arithmetic saves direct numeration of counting, and algebra--in "substituting relations for values--symbolizes and definitively fixes all numerical operations that follow the same rule" (Mach, 1960, p. 583). When numerical operations are symbolized by mechanical operations with symbols, "our brain energy is spared for more important tasks" (Ibid., p. 584), such as discovery or planning. Although overly neural in his explanation, Mach's point about the freeing up of mental capacity by means of making automatic component functions in problem solving is a central theme in cognitive science today.

Whitehead, philosopher and mathematical sage, urged a similar point (1948; see also Cummins, 1969): "by relieving the brain of all unnecessary work, a good notation sets it free to concentrate on more advanced problems, and in effect increases the mental power" (p. 39). He noted that an ancient Greek mathematician would be astonished to learn that today a large proportion of the population can perform the division operation on extremely large numbers (Menninger, 1969). He would be even more astonished to learn that, with calculators, knowledge of long division algorithms is now completely unnecessary. Further arguments about the transformational roles of symbolic notational systems in how mathematical thinking occurs are offered by Cajori (1929 a,b), Grabiner (1974) and, particularly, Kaput (in press).

Although long on insight, Mach and Whitehead lacked a cognitive psychology that explicated the processes through which mathematical thinking could be facilitated and reorganized by means of new technologies. What aspects of mathematical thinking could be freed up, catalyzed, or uncovered by new cognitive technologies? The remainder of this paper is devoted to exploring this central question.

An historical approach is critical so that we can see how time-locked is our current thinking about what it means to think mathematically and to be mathematically educated (cf. Resnick & Resnick, 1977, on comparable historical redefinitions of "literacy" in American education). These questions become all the more fundamental when we realize that, to date, our cognitive and educational research conclusions on "what students can do in mathematics," as tied to age or

Piagetian developmental level, are restricted to the static medium of mathematical thinking with paper and pen[1]. With the dynamic and interactive media provided by computer software, understanding in an intuitive way the interrelationships between graphic, equational, and pictorial representations (traditionally the province of the professional mathematician) becomes more accessible to the software user. Doors to mathematical thinking are opened up, with the result that the "basic findings" of mathematical education will need to be rewritten for the new abodes. Past findings, at the very least, should not constrain our imaginations for what students may be capable of, as we use and create new cognitive technologies for mathematics education.

Transcendent Functions for Cognitive Technologies in Mathematics Education

What strategy shall we choose for thinking about and selecting among cognitive technologies in mathematics education? I will argue for the need to move beyond the familiar cookbooks of 1001 things, in near-random order, that one can do with a computer in mathematics education. Such lists are usually so vast as to be unusable to guide current choice and future development of mathematics educational technologies. Instead, we should seek out high-leverage aspects of information technologies for promoting the development of mathematical thinking skills. I will take the risk of proposing a list of what may be called "transcendent functions" for cognitive technologies in mathematics education.

What is the status of such a list of functions? Certainly not that, if reflected in a piece of software, these functions would provide the necessary and sufficient conditions for promoting mathematical thinking. The thrust of the strategy is more probabilistic in nature; that is, other things being equal, more students are likely to think mathematically when technologies express these functions. A few students will become prodigious mathematical thinkers, whatever the obstacles they have to overcome in their mathematics education.[2] Others will not thrive without richer environments for fostering mathematical thinking. The taxonomy is designed to serve as a heuristic or guide. Assessments of whether it is useful will emerge from empirical research programs, not intuitive conjecture. The list offered builds upon but is not limited to what we know from research in the cognitive sciences,[3] since few tight links between theory and practice can be identified.

Finally, why should we focus on those functions for technologies in mathematics education that are transcendent? There are two major reasons why transcendency of function is important. We would like to know what may be general features of mathematics cognitive technologies, providing generative ideas, so that the technologies do not need to be created from the ground up, mathematics domain by mathematics domain. We would like them to be transcendent in the sense that they apply not only to arithmetic or algebra or calculus, but potentially across a wide array, if not all, of the disciplines of mathematics education--past, present, and future. The transcendent functions of

mathematics cognitive technologies should thus survive changes in the K-12 mathematics curricula, since they exploit general features of what it means to think mathematically, those features at the core of the psychology of mathematics cognition and learning. These functions for technologies in mathematics education should be central regardless of the career emphasis of the student, whether or not college bound. Lessons learned from research and practice with respect to these functions should thus allow productive generalizations.

The transcendent functions to be highlighted are those presumed to have great impact on mathematical thinking. These functions neither begin nor end with the computer. They arise through the culture of teaching in human interactions. The technologies thus only have a role within these contexts of human action and purpose. Nonetheless, interactive media may offer extensions of these critical functions. For any of the specified functions, we can ask: What is special about information technologies that makes the nature or variety of mathematical experience qualitatively different and more likely to precipitate mathematics learning and development?

These functions are by no means independent, nor is it possible to make them so. They define central tendencies with fuzzy boundaries, like concepts in general (Rosch & Mervis, 1975). They are also not presented in order of relative importance. I will illustrate through examples how many recently developed, outstanding mathematics educational technologies incorporate quite a few of the functions. But very few of these programs reflect all of the functions. And only rare examples in classical computer-assisted instruction, where preexisting drill and practice activities have been computerized, reflect any of the functions.

It is important to recognize that one could approach the question of technologies for mathematics education in quite different ways than the one proposed. For example, one can imagine approaches which assume that the dominant role for technology is that of amplifier to give students more practice, more quickly, in applying algorithms that can be carried out faster by computers. I will also not be describing uses of computers for teacher record-keeping or preparing problems for tests or grading tests, although these functions obviously have utility in their own right. But I would not describe them as cognitive technologies.

An alternative perspective of roles for computer technologies in mathematics education is offered by Kelman et al. (1983) in their book, Computers in Teaching Mathematics. They provide an inspiring account of ways software can be used to help create effective environments for student problem solving in mathematics. Their comprehensive book is organized according to traditional software categories and curriculum objectives: computer-assisted instruction, problem solving, computer graphics, applied mathematics, computer science, programming and programming languages. The spirit of their recommendations is in harmony with the sketch to be proposed. The main differences reside in their predominantly curricular rather than cognitive orientation.

My stress on transcendent functions is thus a complementary approach, taking as a starting point the root or foundational psychological processes involved in software that engages mathematical thinking.

I have also leaned heavily in my choice of software illustrations toward those cases that render more overt or manifest their specific loci of support of the seven Purpose or Process functions. Although programming languages, spreadsheets, simulation modeling languages such as MicroDynamo (Addison-Wesley), and symbolic calculators such as muMath (MicroSoft) and TK!Solver (Software Arts) are devices centrally involved in thinking mathematically in an information age, I rarely refer to them as examples. I take for granted the utility and power of these categories of tools in the hands of a person who is committed to problem solving, but the reasons for these powers and utilities have to do with the Functions outlined. For example, Logo graphics programming provides the different mathematical representations of procedural text instructions and the graphics drawing it creates (Function 5); simulation modeling languages and spreadsheets are excellent environments for mathematical exploration (Function 4), since hypothesis-testing and model development-refinement are the central uses of these interactive software tools. But other environments for their use, such as drill and practice on programming language syntax or abstract exercises to write programs to create fibonnaci number series, need not offer much encouragement for mathematical thinking. In other words, the intrinsic value of such tools in helping students think mathematically is not a given. So the stress on Functions remains central.

A Guiding Dichotomy: Purpose and Process Functions for Cognitive Technologies

How can technology support and promote thinking mathematically? In broad strokes, what appear to be the richest loci of potential cognitive and motivational support of technologies for mathematics education?

We can think of two sides to the educational practices of mathematics learning, and ask how software can help. The first is the "personal" side: will the student choose to commit him or herself to learning to think mathematically? Here we highlight motivation and purpose for mathematical thinking. These concepts are sorely neglected in mathematics education (e.g., McLeod, 1985), and may have much to do with the disinterest of girls and minorities in mathematics. The second is the "knowledge" side: will the student be cognitively supported while learning the many problem-solving skills involved in thinking mathematically?

I will assume a two-part perspective on functions for cognitive technologies. First, students are purposive, goal-directed learners who have the will (on any given occasion or over time) to learn to think mathematically or not. Once they have embarked on mathematical thinking, they may be aided in this process by technologies. For

simplicity of exposition, we thus make a major division of function types between those which:

- Promote Purpose: Engage students in thinking mathematically.
- Promote Process Aid them once they do so.

Purpose Functions of Cognitive Technologies

What lies at the heart of cognitive technologies with the functions of making mathematical thinking purposeful and the learner committed to the pursuit of understanding? The fundamental relationship to be supported is a participatory rather than arbitrary link between self and knowledge. The organic self-knowledge relation was central to Dewey's pedagogical writings, and ingredient to Piaget's constructivism: build on the child's interests, desires, and concerns, and on the foundation of the child's world view. But what exactly does this mean?

The key idea of Purpose functions is that they promote the formation of pro-mathematics belief systems in which the student has participation in and ownership of what is learned as a mathematical thinker, rather than serving as an executor of or storage bin for "someone else's mathematics." The implication is that technologies for mathematics education should serve as tools for promoting the student's self-perception as mathematical "agent," as subject or creator of mathematics (Papert, 1972, 1980). For example, Schoenfeld (1985a,b) argues that the belief systems an individual holds can dramatically influence the very possibilities of mathematical education:

> Students abstract a "mathematical world view" both from their experiences with mathematical objects in the real world and from their classroom experiences with mathematics. These perspectives about the ways mathematics works...affect the ways students behave in mathematical situations. They shape what they see in those situations, and what sets of idea, or cognitive resources, they bring to bear on them. (1985b, p. 23)

Although Schoenfeld's focus is broader than the point here, the student's mathematical world view includes the self: What am I in relation to this mathematical behavior I am producing? If the student does not view herself as a mathematical thinker but only as a recipient of the "inert" mathematical knowledge that others possess (Whitehead, 1929), then mathematics education for thinking is going to be problematic because the agent is missing.

In the prototype educational setting, we often erroneously presuppose that we have engaged the child's commitment to learning. But the child rarely sees the significance of the learning; someone else has made all the decisions about scope and sequence, and the lesson for the day. Rather than learning to solve the problems they encounter or a problematic situation that the teacher has brought to their atten-

tion, the students are presented with someone else's knowledge, something someone else found to be useful in the past--but even that fact is rarely conveyed (e.g., how measurement activities were essential to building projects, or making clothing, or numeration systems for trade [McLellan & Dewey, 1895]).

In Dewey's scheme for the logic of inquiry (1933, 1938), the prototypic mathematics fact delivery system leaves out the necessary first step in problem solving: the identification of a problem, the tension that arises between what the student already knows and what she needs to know that drives subsequent problem-solving processes. It is interesting that Polya (1957) also omits this first step. In other respects his phases of problem solving correspond to Dewey's seminal treatment: problem definition, plan creation, plan execution, plan evaluation and reflection for generalization of what can be learned from this episode for the future (cf., Noddings, 1985). Perhaps the expert mathematician takes this first step for granted: "For who could not notice mathematical problems? The world is full of them!" But for the child meeting the formal systems that mathematics offers and the historically accrued problem-solving contexts for which mathematics has been found useful, the first step is a giant one requiring support.

Functions for mathematics educational technologies that support mathematical purpose could be embodied in many ways. Can we say more than was said in Dewey's time about what might be important conditions for promoting mathematical agency? Several component features of mathematical agency appear central:

1. Ownership. Agency is more likely when the student has primary ownership of the problem for which the knowledge is needed (or secondary ownership; that is, identification with the actor in the problem setting in an "as if it were me" simulation). A central pedagogical concern is to find ways to help people "own" their thoughts and the problems through which they will learn. Kaput (1985a) and Papert (1980) have provided suggestive examples from software mathematics discovery environments where the "epistemological context" is redefined: authority for what is known must rest on proof by either the child or the teacher, and does not rest exclusively with teacher and text. Students can offer new problems to be solved and create new knowledge.

2. Self-worth. It is hard to be a mathematical agent if occasions for thinking are viewed by the student as opportunities for failure and diminished self-worth. A great deal rests on the student's self-concept and self-evaluation (Harter, 1985). Research on achievement motivation by Dweck and colleagues (e.g., Dweck & Elliot, 1983) indicates that selection of achievement goals is critically influenced by which of two dominant views of intelligence a student holds. If the child views intelligence as "entity"--a given quantity of stuff that one does or does not have--then the learning events that school arranges are seen as occasions where one either looks smart or fails. In contrast, if the child views intelligence as "incremental,"

then these same learning events are viewed as opportunities for acquiring new understanding. Although little is known about the ontogenesis of the detrimental entity view, it is apparent that this belief can cripple the possibility of mathematical agency, and that software or teaching practices that foster an incremental world view should be sought after. This possibility will be discussed further for the second Purpose function: Social Environments.

3. Knowledge for action. A third condition for promoting mathematical agency is to encourage students to acquire new mathematical knowledge and skills that will make a difference in action, either in their own lives, their future careers, or in solving the real-world problems they will encounter. New knowledge, whether problem-solving skills or mathematical ideas, should empower the child to understand or do something she could not understand or do as well without. That this condition is important is clear from research on the transferability of instructed thinking skills such as memory strategies (e.g., Campione & Brown, 1978), which indicates more transfer of the new skills to other problem settings if one simply explains the benefits of the to-be-learned skill; for example, that more material will be remembered if one learns this strategy.

Technologies for mathematical thinking expressing these Purpose functions should lead students to easily answer the question: What difference does what I have learned make for my acting in my world?

In summary, in characterizing the general category of Purpose functions for cognitive technologies, I have focused on the importance of linking the child-as-agent and the knowledge-acquired for achieving purposive conditions for mathematical thinking, and not on the alleged "motivational value" per se (e.g., Lepper, 1985; Malone, 1981) of mathematics educational technologies. This neglect is intentional. Thinking about technologies as artifacts that cause or induce motivation is inappropriate. Such an orientation has unfortunately led to the extrinsic motivation highlighted in most current learning-game software: bells and whistles are added that serve no function in the student's mathematical thinking. Furthermore, these extrinsic "motivational" features are not pro-agentive in the sense earlier described. With these provisos in mind, I would like to distinguish two major functions of cognitive technologies in facilitating purpose in mathematical thinking: Functional Environments, and Social Environments.

Functional environments for mathematical thinking are those which help elicit motivations for thinking mathematically by providing mathematics activities with purposes beyond "learning mathematics" per se. Whole problems are the focus, in which the mathematics to be learned is necessary to effectively deal with the problem. The mathematics becomes functional, since the technologies are used to motivate mathematical thinking; they prompt the development of mathematical thinking as a means to solving a problem rather than as an end in itself. Systems with this feature help guide students to interpret the world mathematically in a problem-solving context. Associated

curricula are not disembodied from some purpose in actual problem-solving "life" (e.g., Bell, 1979; Lesh, 1981). In other words, the student sees that the mathematics used has a point, and can join in the learning activities that pursue that point. We believe that students will become more involved if they initiate and own a problem, or if they see realistic applications of new knowledge that empower them rather than vague promises of how such knowledge will be useful in their later careers--which we know to be often false.

An example is provided in a 3-stage approach to algebra education using new technologies outlined in the recent Computing and Education Report (Fey, 1984, p. 24). In Stage 1, students begin with "problem situations for which algebra is useful, thus offering the best possible motivation," such as science problems of projectile motion, or nonlinear profit or cost functions. In Stage 2, they learn how to solve such equations by guess-and-test successive approximations--by hand, by graph, and by computer, as well as by means of formal computer tools such as TK!Solver and muMath. In Stage 3, they learn more formal techniques for solving quadratics, such as factoring formulas and number and types of possible roots. Through such a sequence, students do not learn techniques whose promise lies in later applications, but begin by seeing several immediate applications of such mathematics educational activities. It is possible to imagine sequences similar to the one proposed that build from the activities of creating mathematically complex musical and artistic compositions.

Although such functional mathematics learning environments can be created without computers, computers make available new means. For example, software may provide innovative, adventure-like, problem-solving programs for which mathematical thinking is required if the players are to succeed, such as the five programs in Tom Snyder's Search Series (1982), which encourage group problem solving. For example, in Energy Search, students manage an energy factory, collaboratively making interdependent decisions to seek out new sources of energy. Geography Search sets students off on a New World search for the Lost City of Gold, where climate, stars, suns, water depth, and wind direction (and availability of provisions, location of pirates, etc.) must figure in their navigation plans and progress.

In Bank Street's multimedia "The Voyage of the Mimi" Project in Science and Mathematics Education, video, software, and print media weave a narrative tale of a boat odyssey for whale research undertaken by young scientists and their student assistants. Science problems and uses for mathematics and computers emerge and are tackled cooperatively during the group's adventurous voyage (Char, Hawkins, Wootten, Sheingold, & Roberts, 1983). One of the software programs, Rescue Mission (also created by Tom Snyder), simulates navigational instruments (such as radar and a direction finder) used on the Mimi vessel, and the realistic problem-solving event of navigating the boat to save a whale trapped in a fishing net. In order to effectively work together during this software game, students must learn how to plan and keep records of emerging data, work on speed-time-distance problems, reason geometrically, and estimate distances. It is in the context of

needing to do these things that mathematics comes to serve a functional role.

Sunburst Corporation has also created numerous programs that highlight simulations of real-life events, in which students use mathematics skills as aids to planning and problem solving in real-world situations. For example, Survival Mathematics requires mathematical reasoning to solve real-world problems such as best-buy shopping, trip planning, and building construction; and in The Whatsit Corporation, students run a business with one product. The interactive, model-building features of programs like these cannot be well-emulated without computers, and can motivate mathematical thinking in otherwise ambivalent students.

Another way to motivate mathematical thinking is to provide a social context for mathematics problem solving. We may now turn to the corresponding Purpose function that makes this point explicit.

Social environments for mathematical thinking, which promote an interactive social context for discussing, reflecting upon, and/or collaborating in the mathematical thinking necessary to solve a problem: Studies of mathematical problem solving, for example, by Noddings (1985), Pettito (1984a,b), and Schoenfeld (1985b) indicate how useful dialogues can be among different mathematics problem solvers in learning to think mathematically. Small-group dialogues prompt disbelief, challenge, and the need for explicit mathematical argumentation: the group can bring more previous experience to bear on the problem than any individual, and the need for an orderly process to problem solving is highlighted (Noddings, 1985). This focus is also reflected in cooperative learning research in other disciplines of schooling (e.g., Slavin, 1983; Slavin, Sharan, Kagan, Hertz-Lazarowitz, Webb, & Schmuck, 1985; Stodolsky, 1984), and in the new focus in writing instruction that emphasizes thinking-aloud activities (Bereiter & Scardamalia, in press). In the experience of many mathematics educators, the computer can serve as a fundamental mediational tool for promoting dialogue around and collaboration on mathematical problem solving. In mathematical learning, as in writing process activities (Graves & Stuart, 1985; Mehan, Moll, & Riel, 1985), such social settings can open up opportunities for the child to develop a distinctive "voice" in mathematical thinking, and to internalize individually the critical processes that get played out socially in dialogue.

Explicitly facilitating this function with computers in mathematics education has been little utilized to date. The record-keeping and tool functions of software could offer substantial support for collaborative processes in mathematics, as has been suggested for multiple text authoring environments (Brown, 1984b). This function is usually exploited implicitly, as in Logo programming, where students often work together in creating a graphics program, arguing the comparative merits of strategies for solving the mathematics problems that are involved in the programming (Hawkins, Hamolsky, & Heide, 1983; Webb, 1984). The public and revisable nature of the computer screen seems

to enhance student collaborative activities (Hawkins, Sheingold, Gearhart, & Berger, 1982). It also seems likely that a learner's self-worth can emerge in a collaborative context where the part-activities he or she contributes may come to be recognized as "expert" by his or her peers. In Logo programming research, students who had little previous peer support and low self-esteem have come to be recognized as experts in collaborative programming work (Hawkins, Sheingold, & Char, 1984; Papert, Watt, diSessa, & Weir, 1979).

Because mathematics often is a social activity in our world, we should explicitly recognize this and encourage it in mathematics education. In these social contexts, mathematical thinking is a lot more fun for students than they had previously thought possible: it becomes shareable rather than sufferable. In the future, we should be able to provide better tools for collaborating in mathematics problem solving, and work to promote more instructionally relevant peer dialogue around mathematical thinking activities.

One illustrative example is part of "The Voyage of the Mimi" Project in Science and Mathematics Education at Bank Street--a line of software called the Bank Street Lab, developed in conjunction with TERC. It is composed of various kinds of group activities for conducting experiments involving Probe, a hardware device which plugs into the microcomputer and which can measure and graph changes in light, heat, temperature, and sound over time. Students can work together taking measurements and designing and carrying out experiments. Supplementary teacher materials suggest activities where students work as a team to apply mathematical thinking in making scientific discoveries.

Tom Snyder's programs, mentioned above, also offer modes for small groups of students to work cooperatively with or competitively against other student groups.

Process Functions of Cognitive Technologies

A second set of categories of functions for mathematics software is to help students understand and use the different mental activities involved in mathematics thinking. Although our understanding of the psychology of mathematics problem solving and learning is continually evolving, there are five different general categories of Process Functions that can be clearly identified for cognitive technologies in mathematics education. Each provides distinctively important cognitive support for students learning to think mathematically: (1) tools to develop conceptual fluency; (2) tools for mathematical exploration; (3) tools for integrating different mathematics representations; (4) tools for learning how to learn; and (5) tools for learning problem-solving methods. I will briefly define and exemplify each of these categories of functions for mathematics software.

1. Conceptual fluency tools. Fluency tools are those programs that free up the component problem-solving processes by helping students become more fluent in performing the routine tasks in mathematical thinking that could be laborious and counterproductive to

mathematical thinking. Computer technologies can serve as means for promoting such fluency by allowing individually controlled practice on routine tasks with the aim of freeing up a student's mental resources for problem-solving efforts.

There is ample room for debating what these component skills are, particularly in secondary school mathematics (e.g. Fey, 1984; Pollak, 1983), but also in high schools (Maurer, 1984a,b; Usiskin, 1980). There is also little doubt that many software programs routinize irrelevant skills, such as practice on long division algorithms. And the issue is made more complex by the fact that there are many mental functions, such as the component operations of numerical and symbolic calculation, that now can be entirely carried out by mathematics software (e.g., Kunkle & Burch, 1984; Pavelle, Rothstein, & Fitch, 1981; Wilf, 1982; Williams, 1982). What skills and knowledge are required to design the inputs and understand the outputs of such systems?

There is no denying that there are routine mathematical tasks, and one should be able to do them easily to make progress in mathematical thinking. For example, the estimation activities that are at the core of a revised early mathematics curriculum could be made more fluent with information technologies. There are some games involving arithmetic estimation activities that students become proficient at rather quickly (e.g., Pettito, 1984a). It should be noted that this point about routinization is as true at the highest levels of mathematical achievement as it is in primary or secondary school. What is at the farthest reach of mathematical thinking and creativity today is the slog work of tomorrow (Wilder, 1981). Fields keeps advancing, and what is a creative invention at one point, such as Leibniz's calculus or Gauss's development of complex numbers, is likely to become routine and accessible to many others through effective instruction.

The appropriate roles for such fluency tools is the subject of much current debate (Cole, 1985; Mehan et al., 1985; Patterson & Smith, 1985; Resnick, 1985). It is common for schools to erroneously establish a 2-tiered curriculum in which "basic" computational skills are presumed to be prerequisites for more complex, "higher-order" thinking and problem solving. Left to do activities only in the first tier of this curriculum with little motivational significance, many students never get to the mathematical thinking characteristic of the second tier (Cole, 1985). But recent work in writing (in which such a 2-tier approach is common [Mehan et al., 1985; Simmons, in press]) implies that the alleged basic skills can be acquired in the context of more complex activities of mathematical thinking. When a child's conceptual fluency is hampering complex thinking, a functional context for practice to attain fluency is established; this is child-driven in contrast to the 2-tier approach to training the child to some threshold skill level before letting him or her loose to solve problems. Thus, it would appear that insofar as drill-and-practice software for fluency in basic mathematics is concerned, these activities serve the child better as a "fallback" rather than as a "startup" activity.

2. Mathematical exploration tools. Discovery learning has long been an emphasis in mathematics education, particularly in the primary school where manipulatives such as Dienes blocks, Cuisenaire rods, and pattern blocks are emphasized. Today we find computational discovery learning environments for exploring more complex conceptions and mathematical relationships, such as recursion and variables, at a more intuitive level of understanding without abstract symbolic equations.

Logo programming is a paradigm case. The creation of such systems is founded on the assumption that one can come to recognize patterns and make novel discoveries about properties of mathematical systems through self-initiated search in a well-implemented domain of mathematical primitives (Abelson & diSessa, 1981; Papert, 1980). As a result of recent findings which show that students still encounter conceptual difficulties--for example, with recursion and variables in Logo (Hillel & Samurcay, 1985; Kurland & Pea, 1985; Kurland, Clement, Mawby, & Pea, in press; Kurland, Pea, Clement, & Mawby, 1985; Kuspa & Sleeman, 1985; Mawby, in press; Perkins et al., 1985; Perkins & Martin, 1985), there is much debate on how much and what kinds of structure such discovery environments must have in order to be successful. It is an empirical question how well-lit the paths of discovery need to be.

Many software programs now offer early structured exploration environments to help students over some of these difficulties. Two programs that are recommended as preliminary activities to off-the-shelf Logo are Delta Draw (Spinnaker) and Turtle Steps (Holt, Rinehart & Winston). There are several dozen such programs on the market which allow students to use a command language to create designs and explore in systematic ways such plane geometry concepts as angle and variable.

The Geometric Supposers (Sunburst Corporation; for triangles, quadrilaterals, and lines, see Kaput, 1985; Schwartz & Yerushalmy, in press) are striking examples of a new kind of discovery environment. Using these programs, students make conjectures about different mathematical objects in plane geometrical constructions, such as medians, angles, and bisectors. Intended for users from sixth grade and up, they are designed so that users can explore the characteristics of triangles and such concepts as bisector and angle. In this way, students can discover theorems on their own.

The program is an electronic straight-edge and compass. It comes with "building" tools for defining and labeling construction parts (such as the side of a triangle or an angle), and measurement tools for assessing length of lines, degrees of angles, and areas. Most significantly, it will remember a geometric construction that the student makes on a specific object (such as an obtuse triangle) as a procedure (as in Logo) and will allow the student to replay it on new, differently shaped objects (e.g., equilateral triangles). The exciting feature of the environment is that when constructions are made, interesting properties begin to emerge that cry out for testing on other kinds of triangles. And the measurement tools for angles and line distances and for labeling these mathematical entities make such experimentation

easy to follow up on after the student has made a conjecture about something she or he "supposes" to be true about specific types of geometrical construction. Students find this program an exciting entry into empirical geometry (induction), and it can be used to complement classroom work on proofs (deduction). Several classroom students have discovered previously unknown theorems with this discovery tool.

Kaput (1985) has highlighted the major representational breakthroughs in the Supposers: they allow a particular construction to represent a token of a general type of construction, not just the specific construction itself.

3. Representational tools help students develop the languages of mathematical thought by providing links between different representations of mathematical concepts, relationships, and processes. The goal is to help students understand the mapping relations between different media for representing a mathematical problem, and how change in one system has consequences for changes in other systems. By "languages of thought," I refer to different symbolic systems for thinking mathematically, including:

- equations comprised of mathematical symbols (e.g., linear equations);
- visual Cartesian coordinate graphs of functions in two and three dimensions;
- graphic representations of objects (e.g., in place-value subtraction, the use of "bins" of objects representing different types of place units).

This function of cognitive technologies in mathematics education builds upon empirical studies that demonstrate how competency in mathematical problem solving depends in part on facility with thinking in terms of different representational systems during the problem-solving process. Experts can exploit the strengths of the particularities of different representations depending upon the demands of the problem at hand. For example, many relationships can become visually evident from well-designed graphs that are unapparent from descriptions in terms of text, mathematical equations, or tables of data values (Tufte, 1983). One can often gain insight into mathematical relationships by seeing how they appear when graphically depicted rather than represented as symbolic equations (e.g., albegraic functions) (Kaput, 1985, in press).

Interactive technologies allow very interesting operations. They provide the means for intertwining multiple representations (Dickson, 1985) of mathematical concepts and relationships (e.g., graphs and equations; numbers and pictorial representations of objects the numbers represent). These representations enhance the symbolic tools available to the student and their flexible use during problem solving. They also "time compress" mathematical experience, for example,

by allowing many more graphs to be plotted per unit time (Dugdale, 1982), and make possible new kinds of classroom activities involving data collection, display, and analysis (e.g., Kelman et al., 1983: point and function plotting; histograms).

Likely to be of value for learning translation skills between different representational systems (but thus far untested in research) are manipulable, dynamically linked, and simultaneously displayed representations from different symbol systems (Kaput, 1985). For example, students can change a value for a variable in an equation and observe the consequences of this change on the shape of graph for that function. These experiments can be carried out in a motivating gaming context for algebraic equations and graphs with "Green Globs" (in Graphing Equations by Conduit; cf. Dugdale, 1982) and "Algebra Arcade" (Brooks-Cole). In these games on equations and graphs for kindergarten through sixth grade, the player is given Cartesian x-y coordinate axes with 13 green globs randomly distributed on the graph. The players have to type in equations, which the computer graphs; when a graphed equation hits a glob, it explodes and the player's score increases. Students become facile at knowing how changes in the values of equations, such as adding constants or changing factors (e.g., x to 3x), correspond to changes in the shape of the graph (J. J. Kaput, personal communication, November 1985). They discover equation forms for families of graphs, such as ellipses, lines, hyperbolas, and parabolas.

Further examples of dynamically linked representational tools are programs that give <u>visual meaning</u> to operations on algebraic equations (such as adding constants to conditionally equivalent operations). Operations on equations are simultaneously presented with coordinate graphs so that any action on an equation is immediately reflected in the graph shape (Kaput, 1985; Lesh, in press). The student can literally see that doing "arithmetic" (viz., acting on expressions, not equations) does not change graphs.

Another example is provided by software programs in which different representations are exploited in relation to one another for learning place-value subtraction. We can point to Arithmekit (Sybalsky, Burton, & Brown, 1984), Summit (Feurzeig & White, 1984), and Place-Value Place (Interlearn). In each of these cases, number symbols and pictorial representations of objects (and in the case of Summit, synthesized voice) are used in tandem to help students understand how the symbols and operations on them relate to the corresponding pictorial representations and operations. Only Place-Value Place is commercially available. In this program, a calculator displays the addition and subtraction process using three different representations of number values: as a standard number symbol, as a position on a number line, and as a set of proportionally sized objects (apples, bushels of apples, crates of bushels, truckloads of crates). As students add or subtract numbers, all three displays simultaneously change to illustrate the operation. Kaput (1985) describes efforts under way at the Harvard Educational Technology Center to develop

dynamically linked representational tools for ration and proportion reasoning activities (e.g., mpg and mph problems [Schwartz, 1984]).

Programs expressing this function are excellent examples of how the rapid interactivity and new representational tools provided by the computer create a new kind of learning experience. Students can test hypotheses, immediately see their effects, and shape their next hypothesis accordingly, through many cycles--perhaps many more cycles than they would have the patience or span of attention for with noncomputer technologies.

In addition, the use of computers to display graphs and functions in algebra, transformations in geometry, and descriptive statistics, in dynamically interactive ways, would not be possible in noncomputational environments. In the recent NCTM Computers and Mathematics Report (Fey, 1984), the importance of multiple representations in mathematics education is highlighted. In particular, the authors note the rich possibilities for dynamic study of visual concepts such as symmetry, projection, transformation, vectors, and for developing an intuitive sense of shape and relations as they relate to number and more formal concepts.

4. Tools for learning how to learn. This category refers to software programs that can promote "learning by doing" from the details of specific problem-solving experiences, so that students can consolidate what they have learned in episodes of mathematical thinking--to take Dewey's and Polya's final step in problem solving--by reflecting on and evaluating one's work to assess the potential generalization of methods and results (Brown, 1984b). These tools also make possible, in ways to be described, new activities for "learning how to learn"; they leave traces of the student's solution steps while problem solving.

Why should this function provide a more powerful way of learning from experience? First, the problem-solving tracks that a student leaves behind can serve as explicit material for the learner to study, to monitor, and to assess partial solutions to a problem as they are created in order to get at the central problem of controlling their strategic knowledge and activities during problem-solving episodes (Schoenfeld, 1985a, 1985b).

The crucial feature of such systems is that students have access to trace records of their problem-solving processes (e.g., the network of steps in geometry proofs: Geometry Tutor (Boyle & Anderson, 1984); the sequence of operations in algebra equation-solving, as in AlgebraLand (Xerox PARC; J. S. Brown, 1984b). Each provides an externalization of thought processes for inspection and reflection. Let us look at AlgebraLand in some detail.

Consider linear equation solving. While search is not a central concept in algebra instruction today, a central insight of cognitive science is that learning problem-solving skills in mathematics fundamentally involves search--knowledge about when to select what subgoals

and in what sequence. In most classroom instruction in algebra equation solving, the teacher selects the operator to be applied to an equation (e.g., add-to-both-sides) and the student carries out the arithmetic. The pedagogical flaw in this method is that even if the student knows how to execute subgoals (e.g., to do the arithmetic once the Divide operation has been selected), she does not know when to select the various subgoals (Simon, 1980) when solving equations alone.

Originally created several years ago at Xerox PARC by Brown, Roach, and VanLehn, and currently being revised by Foss for work with middle-school students, AlgebraLand is an experimental system for helping students learn algebra from doing problems (Brown, 1984b). Some of the features of the system to be discussed are illustrated in Figure 1.

The task for the student in this figure is to solve the equation for N (shown in the Solve for N window on the figure's right side). Algebraic operators listed in the Basic Operations window on the bottom righthand side (such as Combine-Terms, Add-to-Both-Sides, Distribute), can be selected to apply to the whole equation or to one of its subexpressions. After selecting the operation and deciding where to apply it, the student can execute it. This creates a second algebra expression.

The Record window, upper right, shows the steps taken to a solution. Its left column lists all the intermediate expressions; its right column shows each operation used to transform an expression. The Search Space window records all the solution steps the student has explored. This space is represented graphically as a search tree that displays solution paths with all the backtracking points and problem-solving moves made while trying to solve the equation. In this case, the student took three different approaches to solving the equation, reflected in the three branches from the original equation. Each intermediate expression that resulted from applying the Do-Arithmetic operator appears in boldface for perceptual clarity. AlgebraLand performs all the tactical, algebraic operations and arithmetic calculations. The student has only to select the operator and its scope of application, which effectively eliminates errors in arithmetic or application of operators, and leaves her free for the real mental work of search and operator evaluation.

Operators are also provided for exploring solution paths. There is an Undo operator that returns the equation to its immediately preceding state, and a Goto operator (not on the menu) that returns to any previous state. The student can also back up a solution path by applying the inverse of a forward operator (e.g., selecting Divide after she has just applied Multiply).

Because the windows show every operator used and every state into which the equation was transformed, students have valuable opportunities to learn from specific tracks of their problem solving, and to play with possibilities. They can explore the search paths of their

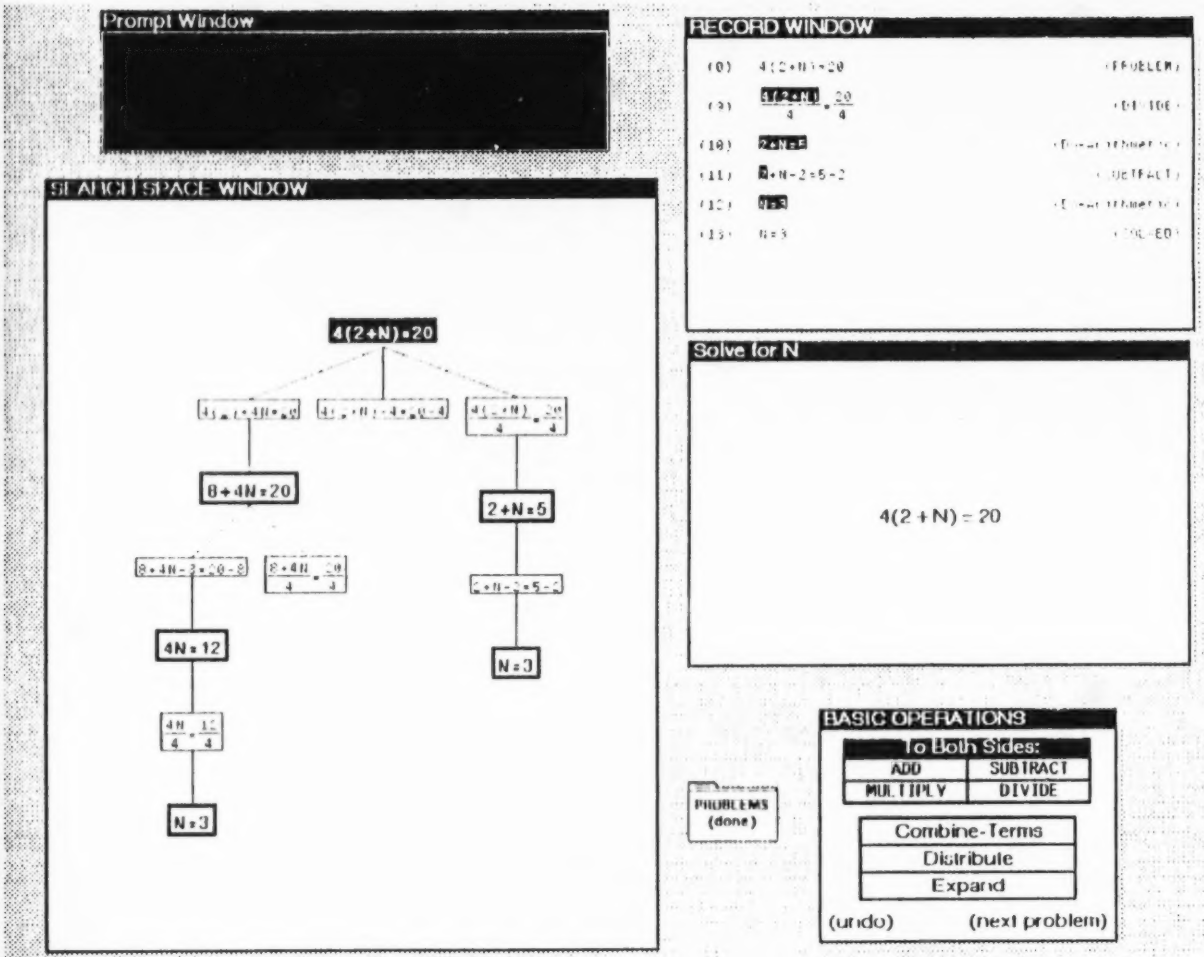

Figure 1
The AlgebraLand screen

solution space, examining branch points where, on one stem, an operation was used that led down an unsuccessful path and, on another stem, trying an operation started down a path toward solution. Then they can study features of the equation at the branch point that may have originally recommended the optimal operation to be used. Guided by this hypothesis for what worked, students can test it out in future equation solving. These learning activities are not possible with traditional methods for learning to solve equations. The cognitive technology offered by AlgebraLand affords new opportunities for different forms and types of learning through problem solving that were not available in static, noncomputer-based symbolic technologies.

In summary, the computer environment AlgebraLand emphasizes a procedure that is diametrically opposed to the traditional instructional method we have described. With AlgebraLand, the student decides when to apply operators, and the computer carries out the mechanical procedures to transform the equation. Students are thus challenged by the problem of searching for and discovering a path of operations that will lead from the problem state to the goal of solving for the unknown. Note how without the graphic availability of the search map, the problem-solving process would be ephemeral, especially when a student's cognitive processes are regularly diverted to applications of operators.

The example of AlgebraLand seems to provide a very rich learning and research environment. But we still know too little about the potential effectiveness of these types of educational instruments. What is important is that new learning problems are introduced: How does the student learn how to "read" and use her traces? How do such learning-how-to-learn skills develop? What ancillary features of these software programs, such as games, will help students understand how to make effective searches in their space of "thinking tracks"?

As yet, such uses of the technology are underrepresented and underdeveloped. But I believe that they could provide a qualitatively new kind of tool for students to learn more effectively from their problem-solving experiences. They also provide a rich data source for teachers' and computers' diagnoses of students' understanding of problem-solving processes and methods.

The provision of explicit traces of a student's problem-solving activities during an episode offers possible entry points for tutorials in problem-solving skills and for intelligent coaching. In a variant on AlgebraLand, students could be prompted to check the operation (e.g., factoring) that they have been using during equation solving activities in order to determine, for example, if they have used that operator in a way that has led to a more complex rather than a simpler equation.

Along the same lines, the information available through such traces provides the data for modeling what a student understands, based on her interactions with a computer in a specific mathematics problem-solving domain. As experience with current prototypes such as the

arithmetic game WEST indicates (Burton & Brown, 1979), such systems allow for coaching dialogues that are sensitive to the student's developmental level, and are quite subtle in how they coach. Unlike some intelligent tutoring systems, like the Geometry Tutor (Boyle & Anderson, 1984) and the Lisp Tutor (Anderson & Reiser, 1985), they don't correct the student after every suboptimal move. It is not yet apparent how broadly this approach to student misconception modeling and coaching can be applied. It is important to emphasize that at the current time all the programs that exemplify this category of function run on minicomputers, and have not been developed commercially for schools.

5. Tools for learning problem-solving methods. This category of tools promotes reasoning strategies for mathematical problem solving. Recent work on the development of mathematical thinking highlights the importance of reasoning strategies. Individuals who have more expertise in approaching problem-solving activities in mathematics possess, in addition to knowledge of mathematics facts and algorithmic procedures, strategies that guide their work on more difficult problems. Such heuristics, well known from the work of Polya (1957) and modern studies of mathematics problem solving (e.g., Silver, 1985), include drawing and annotating diagrams, and exploiting related problems (Schoenfeld, 1985a).

As far as I know, there are currently few examples of mathematics educational technologies that are designed to help students develop the kind of problem-solving heuristics discussed above. One prominent example is WUMPUS, a fantasy computer game in which the player must hunt and slay the vicious Wumpus to avoid deadly pitfalls (Yob, 1975). Goldstein and colleagues (e.g., Goldstein, 1979) created artificial intelligence programs to coach students in acquiring the reasoning strategies in logic, probability, decision analysis, and geometry needed for skillful Wumpus hunting. The Wumpus coaches, which are minicomputer programs, have not been evaluated in educational settings, and the issue of transfer of skills acquired by students beyond this game setting is not emphasized.

Sunburst Corporation (1985) promotes a "problem solving matrix" that is indexed to the software they sell for schools, so that (allegedly) the teacher can know what problem-solving skills and strategies (e.g., binary search) the student will learn by using the program. However, these strategies are not explicitly taught by the software. Furthermore, the scheme erroneously presupposes that one can identify, a priori, what problem-solving skills all students will use at all times in working with a specific program. But the problem-solving processes or component thinking skills a student will use in working with a software program will change with cognitive development. Utilized skills will also vary across individuals, due to cognitive style and other variables.

Notwithstanding these reservations and if sufficient attention were devoted to the effort, current tools in artificial intelligence could be used to create learning environments in which the application of a

mathematics problem-solving heuristic, or set of heuristics, could be exemplified for large numbers of problems. Students could explore these applications and then be offered transfer problems that assess whether they have induced how the heuristic works in a manner sufficient to "carry on" (Wittgenstein, 1956). If not, the system could offer various levels of coaching, or different layers of hints, that would lead to a modeling example for that specific problem if the student wanted it.

Synthesis of Functions: The Need for Interdisciplinary Research on Cognitive Technologies for Mathematics Education

I would like to close this discussion of functions for cognitive technologies in mathematics education by making a plea for interdisciplinary, classroom-based research involving mathematics educators, cognitive scientists, software makers, and mathematicians. We know very little about the educational impact, actual or possible, of different technological applications in mathematics education. We cannot predict how the role of the teacher may change when such technologies are pervasively used, or in what new ways teachers will need to be trained to help students take control of their own mathematical thinking and learning with these technologies. It seems likely that, as the teaching of routine computation algorithms disappears with these new technologies, teachers' jobs will become more intellectually challenging, their activities more like mathematicians'. How will teacher training institutions need to change? We face a plethora of unknowns. Yet such information will be necessary for critically informed and effective uses of cognitive technology by mathematics educators. Without empirical testing of exploratory new instructional curricula that embody these various functions, we will have little idea whether these various functions for cognitive technologies actually promote mathematical thinking.

Coda

These are exciting times for learning mathematics and for shaping the future of mathematical thinking through new technologies. Mathematics was a dreary subject for many of us when we were in school, but mathematical thinking is now often learned through problem-solving activities that embed the mechanical aspects of mathematics in interesting materials. And the puritanical attitude that the mind is a muscle to be exercised by mechanical repetition is giving way to a richer view of the creative, exploring mind which can be nurtured and guided to discover and learn through meaningful problem-solving activities. If we can infuse life into mathematical learning tools and integrate supports for the personal and the knowledge side of mathematical thinking, then perhaps each child will realize how the powerful abstractions of mathematics offer personal power. In such a utopia, learning mathematics is but one more way of learning how to think and to define one's personal voice in the world.

It is hoped that the transcendent functions proposed for cognitive technologies in mathematics education may serve others as useful

heuristics in terms of designing initiatives in building new generations of cognitively meaningful learning and teaching tools for mathematics thinking. Others are bound to offer a different set of transcendent functions, but clarification of the issues gained through such debate will contribute to the fundamental goal--using cognitive technologies in mathematics education to most effectively prepare students for the complexities of mathematical thinking. Since the major sources of help in this enterprise will be the creators and consumers of educational software, it is up to us to provide heuristics to help guide their thinking about what--from a psychological perspective--the technologies could and should offer.

Notes

[1]This argument is developed more fully with respect to cognitive development in general with new technologies in Pea (1985).

[2]It is more commonly true that prodigious mathematical thinkers have had a remarkable coalescence of supportive environmental conditions for their learning activities, including suitable models, rich resource environment of learning materials, community of peers, and private tutoring (e.g., Feldman, 1980).

[3]This situation is the rule in theory-practice relations in education (Champagne & Chaiklin, 1985; Suppes, 1978). For this reason, I have recently proposed (Pea, in press) the need for an activist research paradigm in educational technology, with the goal of simultaneously creating and studying changes in the processes and outcomes of human learning with new cognitive and educational tools.

References

Abelson, H., & diSessa, A. A. (1981). Turtle geometry: The computer as medium for exploring mathematics. Cambridge, MA: MIT Press.

Anderson, J. R., & Reiser, B. (1985). The LISP tutor. Byte, 10, 159-175.

Bell, M. S. (1979). Applied problem solving as a school emphasis: An assessment and some recommendations. In R. Lesh, D. Mierkiewicz, & M. G. Kantowski (Eds.), Applied mathematical problem solving. Columbus, OH: ERIC Clearinghouse of Science, Mathematics, and Environmental Education.

Bereiter, C., & Scardamalia, M. (in press). The psychology of written composition. Hillsdale, NJ: Erlbaum.

Boyle, C. F., & Anderson, J. R. (1984). Acquisition and automated instruction of geometry proof skills. Paper presented at the meeting of the American Educational Research Association, New Orleans, LA.

Brown, J. S. (1982). Learning-by-doing revisited for electronic learning environments. In M. A. White (Ed.), The future of electronic learning. Hillsdale, NJ: Erlbaum.

Brown, J. S. (1984a). Idea amplifiers: New kinds of electronic learning environments. Palo Alto, CA: Xerox PARC, Intelligent Systems Laboratory.

Brown, J. S. (1984b). Process versus product: A perspective on tools for communal and informal electronic learning. In S. Newman & E. Poor (Eds.), Report from the Learning Lab: Education in the electronic age. New York: Educational Broadcasting Corporation, WNET.

Bruner, J. S. (1966). On cognitive growth. In J. S. Bruner, R. R. Olver, & P. M. Greenfield (Eds.), Studies in cognitive growth. [City, Publisher?]

Burton, R. R., & Brown, J. S. (1979). An investigation of computer coaching for informal learning activities. International Journal of Man-Machine Studies, 11, 5-24.

Cajori, F. (1929a). A history of mathematical notations: Notations in elementary mathematics (Vol. 1). La Salle, IL: Open Court.

Cajori, F. (1929b). A history of mathematical notations: Notations mainly in higher mathematics (Vol. 2). La Salle, IL: Open Court.

Campione, J. C., & Brown, A. L. (1978). Training general metacognitive skills in retarded children. In M. Grunberg, P. E. Morris, & R. N. Sykes (Eds.), Practical aspects of memory. New York: Academic Press.

Cassirer, E. (1910). Substance and function and Einstein's theory of relativity. Mineola, NY: Dover.

Cassirer, E. (1944). An essay on man: An introduction to a philosophy of human culture. New Haven, CT: Yale University Press.

Cassirer, E. (1957). The philosophy of symbolic forms: The phenomenology of knowledge (Vol. 3). New Haven, CT: Yale University Press.

Champagne, A. B., & Chaiklin, S. (1985). Interdisciplinary research in science learning: Great prospects and dim future. Paper presented at the meeting of the American Educational Research Association. [Where?]

Char, C., Hawkins, J., Wootten, J., Sheingold, K., & Roberts, T. (1983). "The Voyage of the Mimi": Classroom case studies of software, video, and print materials. New York: Bank Street College of Education, Center for Children and Technology.

Cole, M. (1985). Non-cognitive factors in education: Subcommittee report. Washington, DC: National Research Council (NRC) Commission on Behavioral and Social Sciences and Education.

Cole, M., & Griffin, P. (1980). Cultural amplifiers reconsidered. In D. R. Olson (Ed.), The social foundations of language and thought: Essays in honor of Jerome S. Bruner. New York: Norton, 343-364.

Conference Board of the Mathematical Sciences. (1983). The mathematical sciences curriculum K-12: What is still fundamental and what is not. Washington, DC: Author.

Cummins, K. (1969). Equations and the ways they were written. In Historical topics for the mathematics classroom: Thirty-First Yearbook of the National Council of Teachers of Mathematics (pp. 260-263). Washington, DC: The Council.

Dewey, J. (1933). How we think. Chicago: Henry Regnery.

Dewey, J. (1938). LOGIC: The theory of inquiry. New York: Henry Holt.

Dickson, P. (1985). Thought-provoking software: Juxtaposing symbol systems. Educational Researcher, 14, 30-38.

Dugdale, S. (1982, March). Green Globs: A microcomputer application for the graphing of equations. Mathematics Teacher, 208-214.

Dweck, C. S., & Elliot, E. S. (1983). Achievement motivation. In E. M. Hetherington (Ed.), Socialization, personality, and social development (Vol. IV) (pp. 643-691), of P. H. Mussen (Ed.), Handbook of child psychology (4th ed.). New York: Wiley.

Elgarten, G. H., Posamentier, A. S., & Moresh, S. E. (1983). Using computers in mathematics. Menlo Park, CA: Addison-Wesley.

Feldman, D. H. (1980). Beyond universals in cognitive development. Norwood, NJ: Ablex.

Feurzeig, W., & White, B. (1984). An articulate instructional system for teaching arithmetic problems (Tech. Rep.). Cambridge, MA: Bolt Beranek & Newman.

Fey, J. T. (1984). (Ed.). Computing and mathematics: The impact on secondary school curricula. College Park, MD: National Council of Teachers of Mathematics.

Fey, J. T. (Ed.), with Atchison, W. F., Good, R. A., Heid, M. K., Johnson, J., Kantowski, M. G., & Rosen, L. P. (1984). Computing and mathematics: The impact on secondary school curricula. College Park, MD: National Council of Teachers of Mathematics.

Goldstein, I. (1979). The genetic epistemology of rule systems. International Journal of Man-Machine Studies, 11, 51-57.

Goodman, N. (1976). Languages of art (rev. ed.). Amherst, MA: University of Massachusetts Press.

Goody, J. (1977). The domestication of the savage mind. New York: Cambridge University Press.

Gould, S. J. (1980). Shades of Lamarck. In The panda's thumb: More reflections in natural history. New York: Norton.

Grabiner, J. V. (1974). Is mathematical truth time-dependent? American Mathematical Monthly, 81, 354-365.

Graves, D. H., & Stuart, V. (1985). Write from the start: Tapping your child's natural writing ability. New York: Dutton.

Greenfield, P. (1972). Oral and written language: The consequences for cognitive development in Africa, the United States, and England. Language and Speech, 15, 169-178.

Greenfield, P. M., & Bruner, J. S. (1969). Culture and cognitive growth (rev. ed.). In D. A. Goslin (Ed.), Handbook of socialization theory and research. Chicago: Rand McNally.

Harter, S. (1985). Competence as a dimension of self-evaluation: Toward a comprehensive model of self-worth. In R. Leahy (Ed.), The development of the self. New York: Academic Press.

Hawkins, J., Hamolsky, M., & Heide, P. (1983). Paired problem solving in a computer context (Tech. Rep. No. 33). New York: Bank Street College of Education, Center for Children and Technology.

Hawkins, J., Sheingold, K., Gearhart, M., & Berger, C. (1982). Microcomputers in schools: Impact on the social life of elementary classrooms. Journal of Applied Developmental Psychology, 3, 361-373.

Hillel, J., & Samurcay, R. (1985). Analysis of a LOGO environment for learning the concept of procedures with variable (Tech. Rep.). Montreal, Canada: Concordia University, Mathematics Department.

Kaput, J. J. (1985a). Information technology and mathematics: Opening new representational windows. Paper presented at National Conference on Using Computers to Teach Complex Thinking, National Academy of Sciences, Washington, DC.

Kaput, J. J. (in press). Towards a theory of symbol use in mathematics. In C. Janvier (Ed.), Problems of representation and translation in mathematics. Hillsdale, NJ: Erlbaum.

Kelman, P., Bardige, A., Choate, J., Hanify, G., Richards, J., Roberts, N., Walters, J., & Tornrose, M. K. (1983). Computers in teaching mathematics. Reading, MA: Addison-Wesley.

Kessel, F. S., & Siegel, A. W. (Eds.). (1983). The child and other cultural inventions. New York: Praeger.

Kessen, W. (1979). The American child and other cultural inventions. American Psychologist, 34, 815-820.

Kline, M. (1972). Mathematical thought from ancient to modern times. Oxford, England: Oxford, England University Press.

Kunkle, D., & Burch, C. I. Jr. (1984). Symbolic computer algebra: The classroom computer takes a quantum leap. Mathematics Teacher, 209-214.

Kurland, D. M., Clement, C., Mawby, R., & Pea, R. D. (in press). Mapping the cognitive demands of learning to program. In J. Bishop, J. Lochhead, & D. N. Perkins (Eds.), Thinking. Hillsdale, NJ: Erlbaum.

Kurland, D. M., & Pea, R. D. (1985). Children's mental models of recursive Logo programs. Journal of Educational Computing Research, 1, 235-243.

Kurland, D. M., Pea, R. D., Clement, C., & Mawby, R. (1985). A study of programming ability and thinking skills in high school students (Tech. Rep.). New York: Bank Street College of Education, Center for Children and Technology.

Kuspa, L., & Sleeman, D. (1985). Novice Logo errors. Unpublished manuscript. Stanford University, School of Education and Department of Computer Science, Stanford, CA.

Lepper, M. R. (1985). Microcomputers in education: Motivational and social issues. American Psychologist, 40, 1-18.

Lesh, R. (1981). Applied mathematical problem solving. Educational Studies in Mathematics, 12(2), 235-265.

Lesh, R. (in press). The evolution of problem representations in the presence of powerful cultural amplifiers. In C. Janvier (Ed.), Problems of representation in mathematics learning and problem solving. Hillsdale, NJ: Erlbaum.

Levin, J. (1984). Place-Value Place. Cardiff-by-the-Sea, CA: Inter-Learn.

Luria, A. R. (1976). Cognitive development: Its cultural and social foundations. Cambridge, MA: Harvard University Press.

Luria, A. R. (1979). The making of mind: A personal account of Soviet psychology. Cambridge, MA: Harvard University Press.

Mach, E. (1960). The science of mechanics: A critical and historical account of its development (6th ed). LaSalle, ILL: Open Court. (Originally published, 1893)

McLellan, J. A., & Dewey, J. (1895). The psychology of number: And its applications to methods of teaching arithmetic. New York: Appleton.

Malone, T. W. (1981). Toward a theory of intrinsically motivating instruction. Cognitive Science, 4, 333-369.

Maurer, S. B. (1984a). College entrance mathematics in the year 2000. Mathematics Teacher, 77, 422-428.

Maurer, S. B. (1984b). The effects of a new college mathematics curriculum on high school mathematics. In A. Ralston & G. S. Young (Eds.), The future of college mathematics. New York: Springer-Verlag.

Mawby, R. (in press). Proficiency conditions for the development of thinking skills through programming. In J. Bishop, J. Lochhead, & D. N. Perkins (Eds.), Thinking. Hillsdale, NJ: Erlbaum.

McLeod, D. B. (1985). Affective issues in research on teaching mathematical problem solving. In E.A. Silver (Ed.), Teaching and learning mathematical problem solving: Multiple research perspectives (pp. 267-279). Hillsdale, NJ: Erlbaum.

Mehan, H., Moll, L., & Riel, M. (1984). A quasi-experiment in guided change (Grant #NIE 6-83-0027). Washington, DC: National Institute of Education.

Menninger, K. (1969). Number words and number symbols: A cultural history of numbers. Cambridge, MA: MIT Press.

National Science Board Commission on Precollege Education in Mathematics, Science, and Technology. (1983). Educating Americans for the 21st century. Washington, DC: National Science Foundation.

Noddings, N. (1985). Small groups as a setting for research on mathematical problem solving. In E. A. Silver (Ed.), Teaching and learning mathematical problem solving: Multiple research perspectives (pp. 345-359). Hillsdale, NJ: Erlbaum.

Olson, D. R. (1976). Culture, technology and intellect. In L. B. Resnick (Ed.), The nature of intelligence. Hillsdale, NJ: Erlbaum.

Olson, D. R. (1977). From utterance to text: the bias of language in speech and writing. Harvard Educational Review, 47, 257-281.

Olson, D. R. (1985). Computers as tools of the intellect. Educational Researcher, 14, 5-8.

Olson, D. R., & Bruner, J. S. (1974). Learning through experience and learning through media. In D. R. Olson (Ed.), Media and symbols: The forms of expression, communication, and education. Chicago: University of Chicago Press.

Ong, W. J. (1982). Orality and literacy: The technologizing of the word. New York: Methuen.

Papert, S. (1972). Teaching children to be mathematicians versus teaching about mathematics. International Journal for Mathematical Education, Science, and Technology, 3, 249-262.

Papert, S. (1980). Mindstorms: Children, computers, and powerful ideas. New York: Basic Books.

Papert, S., Watt, D., diSessa, A., & Weir, S. (1979). An assessment and documentation of a children's computer laboratory. Final Report of the Brookline LOGO Project. Cambridge, MA: MIT, Department of Artificial Intelligence.

Patterson, J. H., & Smith, M. S. (1985, September). The role of computers in higher order thinking (Grant #NIE-84-0008). Washington, DC: National Institute of Education.

Pavelle, R., Rothstein, M., & Fitch, J. (1981). Computer algebra. Scientific American, 245, 136-152.

Pea, R. D. (1985). Integrating human and computer intelligence. In E. Klein (Ed.), New directions for child development: Children and Computers (pp. 75-96). San Francisco: Jossey-Bass.

Pea, R. D. (in press). Beyond amplification: Using the computer to reorganize mental functioning. Educational Psychologist.

Pea, R. D., & Kurland, D. M. (1984). On the cognitive effects of learning computer programming. New Ideas in Psychology, 2, 137-168.

Perkins, D. N., Hancock, C., Hobbs, R., Martin, F., & Simmons, R. (1985). Conditions of learning in novice programmers (Tech. Rep.). Cambridge, MA: Harvard Graduate School of Education, Educational Technology Center.

Perkins, D. N., & Martin, F. (1985). Fragile knowledge and neglected strategies in novice programmers (Tech. Rep.). Cambridge, MA: Harvard Graduate School of Education, Educational Technology Center.

Pettito, A. L. (1984a). Beyond drill and practice: Estimation and collaboration in a microcomputer environment. Paper presented at the meeting of the American Educational Research Association, New Orleans, LA.

Pettito, A. L. (1984b). Discussant's comments on misconceptions and comments on important issues for research in science and mathematics education. Paper presented at the meeting of the American Educational Research Association, New Orleans, LA.

Pollak, H. (1981). Interview. In L. A. Steen & D. J. Albers (Eds.), Teaching teachers, teaching students. Boston: Birkhauser.

Pollak, H. (1983). The mathematics curriculum: What is still fundamental and what is not. Washington, DC: Conference Board of the Mathematical Sciences.

Polya, G. (1957). How to solve it. New York: Doubleday-Anchor.

Resnick, D. P., & Resnick, L. B. (1977). The nature of literacy: An historical explanation. Harvard Educational Review, 47 , 370-385.

Resnick, L. B. (1985). Education and learning to think: Subcommittee report. Washington, DC: National Research Council Commission on Behavioral and Social Sciences and Education.

Rosch, E., & Mervis, C. (1975). Family resemblances: Studies in the internal structure of categories. Cognitive Psychology, 7, 573-605.

Schoenfeld, A. (1985a). Mathematical problem solving. New York: Academic Press.

Schoenfeld, A. (1985b). Metacognitive and epistemological issues in mathematical understanding. In E. A. Silver (Ed.), Teaching and learning mathematical problem solving (pp. 361-379). Hillsdale, NJ: Erlbaum.

Schwartz, J. (1984). An empirical determination of children's word problem difficulties: Two studies and a prospectus for further research (Tech. Rep.). Cambridge, MA: Harvard Graduate School of Education, Educational Technology Center.

Schwartz, J., & Yerushalmy, M. (1985). The Geometric Supposers. Pleasantville, NY: Sunburst Communications.

Schwartz, J., & Yerushalmy, M. (in press). The Geometric Supposer: The computer as an intellectual prosthetic for the making of conjectures. The College Mathematics Journal.

Scribner, S., & Cole, M. (1981). The psychology of literacy. Cambridge, MA: Harvard University Press.

Segal, J. W., Chipman, S. F., & Glaser, R. (Eds.). (1985). Thinking and learning skills: Relating instruction to basic research (Vol. 1). Hillsdale, NJ: Erlbaum.

Sheingold, K., Hawkins, J., & Char, C. (1984). "I'm the thinkist, you're the typist": The interaction of technology and the social life of classrooms. Journal of Social Issues, 40, 49-61.

Silver, E. A. (Ed.). (1985). Teaching and learning mathematical problem solving: Multiple research perspectives. Hillsdale, NJ: Erlbaum.

Simmons, W. (in press). Beyond basic literacy skills. In R.D. Pea & K. Sheingold (Eds.), Children and microcomputers: Theory, research, and development from Bank Street's Center for Children and Technology. Norwood, NJ: Ablex.

Simon, H. A. (1980). Problem solving and education. In D. Tuma & F. Reif (Eds.), Problem solving and education. Hillsdale, NJ: Erlbaum.

Slavin, R. (1983). Cooperative learning. New York: Longman.

Slavin, R., Sharan, S., Kagan, S., Hertz-Lazarowitz, R., Webb, C., & Schmuck, R. (1985). Learning to cooperate, cooperating to learn. New York: Plenum.

Sleeman, D., & Brown, J. S. (Eds.). (1982). Intelligent tutoring systems. New York: Academic Press.

Snyder, T. (1982). Search series. New York: McGraw-Hill.

Stodolsky, S. (1984). Frameworks for studying instructional processes in peer work-groups. In P. L. Peterson, L. C. Wilkinson, & M. Hallihan (Eds.), The social context of instruction. Orlando, FL: Academic Press.

Suppes, P. (Ed.). (1978). Impact of research on education: Some case studies. Washington, DC: National Academy of Education.

Sybalsky, R., Burton, R., & Brown, J. S. (1984). Arithmekit. Palo Alto, CA: Xerox PARC.

Tufte, E. R. (1983). The visual display of quantitative information. Cheshire, CT: Graphics Press.

Usiskin, Z. (1980). What should not be in the algebra and geometry curricula of average college-bound students? Mathematics Teacher, 73, 413-424.

Vygotsky, L. S. (1962). Thought and language. Cambridge, MA: MIT Press.

Vygotsky, L. S. (1978). Mind in society: The development of the higher psychological processes. Cambridge, MA: Harvard University Press.

Wartofsky, M. (1983). The child's construction of the world and the world's constuction of the child: From historical epistemology to historical psychology. In F. S. Kessel & A. W. Sigel (Eds.), The child and other cultural inventions. New York: Praeger.

Webb, N. M. (1984). Microcomputer learning in small groups: Cognitive requirements and group processes. Journal of Educational Psychology, 76, 1076-1088.

White, S. (1983). Psychology as a moral science. In F. S. Kessel & A. W. Sigel (Eds.), The child and other cultural inventions. New York: Praeger.

Whitehead, A. N. (1929). The aims of education. New York: Macmillan.

Whitehead, A. N. (1948). An introduction to mathematics. Oxford, England: Oxford, England University Press.

Wilder, R. L. (1981). Mathematics as a cultural system. New York: Pergamon.

Wilf, H. S. (1982). The disk with the college education. American Mathematical Monthly, 89, 4-8.

Williams, G. (1982, October). Software Arts' TK!Solver. Byte, 360-376.

Wittgenstein, L. (1956). Remarks on the foundations of mathematics. Oxford, England: Blackwell.

Yob, G. (1975). Hunt the Wumpus. Creative Computing, 51-54.